TAKING IT FURTHER

If you have come this far, you are more than a recreational puzzle solver; you are looking for deeper connections and patterns, which is just what a professional mathematician does. After you have looked over these remarks, and perhaps some of the references, please come and visit MSRI, the AMS (American Mathematical Society), and the MAA (Mathematical Association of America).

Thinking a little bit about the original problem, if we go to a larger grid, we can see a pattern emerging.

The number of squares lying within a six-by-six grid, say, is the sum of the first six square numbers, $1 + 4 + 9 + 16 + 25 + 36 = 91$, and the number of squares hidden in a ten-by-ten array is the sum of the first ten square numbers: $1 + 4 + 9 + \cdots + 81 + 100 = 245$.

Finding these specific numbers is gratifying, but we can say a lot more!

Let's think about a new and apparently unrelated problem.

> **I have five friends: Ayeesha, Ben, Clay, Duane, and Emma. I may invite only two over for a party. How many choices of pairs do I have?**

To answer this, one can simply begin listing pairs of friends. If we do this in a systematic manner, a pattern emerges. Using simply the first letters of their names we have:

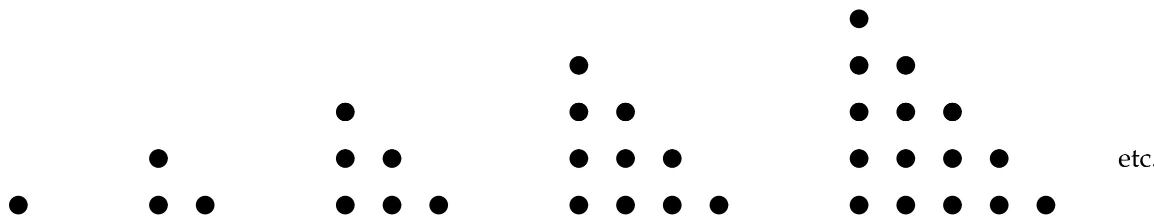
AB
AC BC
AD BD CD
AE BE CE DE

yielding $4 + 3 + 2 + 1 = 10$ possible pairs. Why don't we need to include combinations like **AA** or **CA**? What if there is a sixth friend, Frank?

In the same way one can show that from among six people one can select $5+4+3+2+1 = 15$ possible pairs of friends, and from ten people, $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ possible pairs. The numbers that arise this way:

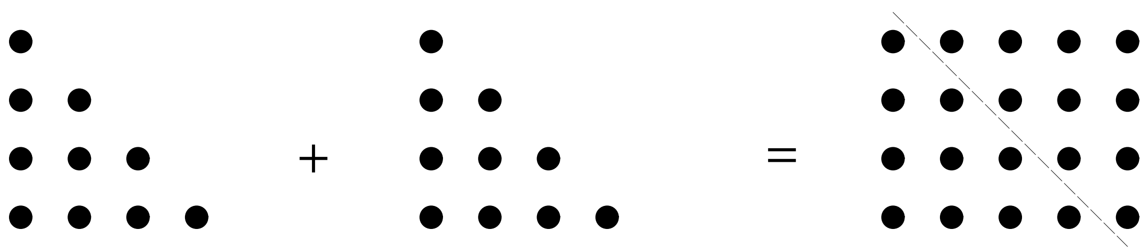
$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$
$$1 + 2 + 3 + 4 + 5 = 15$$
$$\vdots$$
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

are called *triangular numbers* because of the triangular designs that appear when you list all possible pairs of friends. Here we simply use a dot to represent a pair.



1    $1+2=3$ $1+2+3=6$ $1+2+3+4=10$ $1+2+3+4+5=15$

Two copies of any particular triangular design combine to make a rectangle one unit longer than its width:



$1+2+3+4$ $1+2+3+4$ 4×5

Thus we see that $2\times(1+2+3+4)=4\times 5$, or, put another way, $1+2+3+4=(4\times 5)/2$. In general, this technique shows that the n-th triangular number is given by the formula $1+2+3+4+\ldots+(n-1)+n=n\times(n+1)/2$ and this is the number of possible ways to choose a pair of people from among $n+1$ friends. For example, there are $(100)(101)/2=5050$ possible pairs of friends among 100 people.

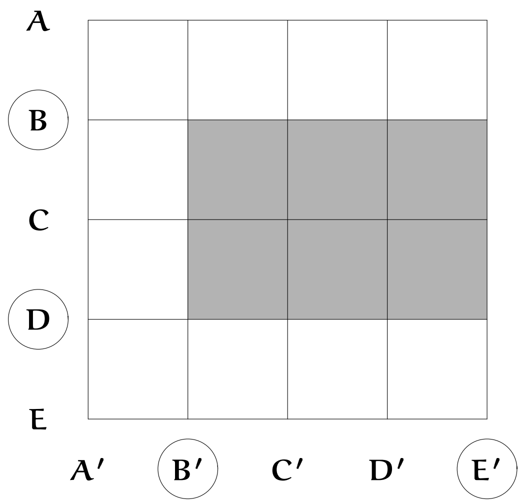
You might recognize the number $(n+1)n/2$ as the number of combinations of $n+1$ things (say the numbers $1,2,3,\ldots,n,n+1$) taken 2 at a time, and you might want to consider why this is so.

(This number is often written $C(n+1,2)$ or $\binom{n+1}{2}$.)

BACK TO RECTANGLES

Let's return to the problem of counting the number of rectangles in a four-by-four array of squares. This time, let's list the names of our five friends to the left of the grid and list the names of a second set of five friends below the grid. Of course the count of friends is one more than the count of squares reading across a row or down a column.

Each choice of a pair of friends from the list on the left, along with a choice of pair of friends from the list below the array, defines a rectangle in the grid. Conversely, any rectangle in the grid is defined by a pair of friends from each list. As we have seen there are ten possible pairs from the list of five people on the left, and ten possible pairs for the list below the grid, yielding 10×10 possible choices in all. Thus there are 100 rectangles in the grid!



This same reasoning shows then that in a 100×100 grid of squares, for example, there will be 101 "friends" along each side, and so $5050 \times 5050 = 25{,}502{,}500$ different rectangles. It would be impossible to count these by hand. The power of the mind and systematic reasoning wins over the brute force approach!

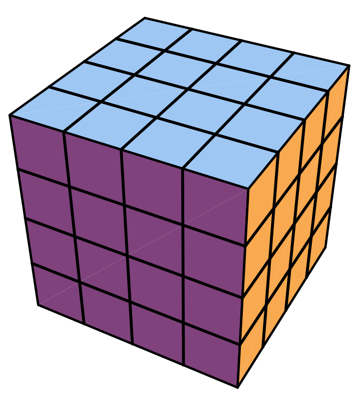
Notice that this argument also applies to non-square grids. For example, we can now say with assuredness that, in a 16×43 grid of squares, there lie

$$(1+2+3+\cdots+16) \times (1+2+3+\cdots+43) = \frac{16 \times 17}{2} \times \frac{43 \times 44}{2} = 136 \times 946 = 128{,}656$$

different rectangles.

TAKING IT UP A DIMENSION

Consider the number of cubes of any kind in a (three-dimensional) $4 \times 4 \times 4$ array of cubes:



One can solve this problem by counting the number of vertices in the cubical array that could serve as a top left *front* corner of an inner cube. For example, only 1 vertex can serve as the top left corner of a (the) $4 \times 4 \times 4$ cube, 8 vertices serve as top left corner of a $3 \times 3 \times 3$ cube (can you see this?), 27 vertices serve as the top left front corner of a $2 \times 2 \times 2$ cube, and 64 vertices as top left front corners of a $1 \times 1 \times 1$ cube. Notice that $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, and $64 = 4^3$ are the first four "cube numbers" and $1 + 8 + 27 + 64 = 100$.

Is it a coincidence that the number of cubes in the three-dimensional $4 \times 4 \times 4$ array is the same as the number of rectangles in a two-dimensional grid? How does the count of cubes in an array compare to the count of rectangles in an $n \times n$ grid for different values of n?

FURTHER READING

To learn more about triangular, square, and other "figurate numbers" have a look at *The Book of Numbers*, by John H. Conway and Richard K. Guy (Springer-Verlag, New York, 1996). To find formulas for the sum of squares, cubes, and higher-power numbers consider the article "A dozen questions about the sums of powers" by James Tanton, in the September 2003 issue of *Math Horizons* (which you can get from the website of the Mathematical Association of America).